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A NONLINEAR HILLE-YOSIDA-PHILLIPS THEOREM

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Let X denote a real or complex Banach space. By a nonexpansive semi-group, we will mean a strongly continuous semi-group $\{T(t);\ t \geq 0\}$ of nonexpansive transformations from X into X. The strong infinitesimal generator A of such a semi-group $\{T(t);\ t \geq 0\}$ is defined by $Ax = \lim_{h \to 0} A_h x$, with domain consisting of all x for which this limit exists, where $A_h x = (1/h)[G(h)x - x]$ for x in X and $h > 0$. The weak infinitesimal generator is defined similarly, using the weak limit in place of the strong limit. We give a necessary and sufficient condition, in the case that X is a Hilbert space, that a densely defined operator in X be the strong infinitesimal generator of a nonexpansive semi-group. This may be considered a nonlinear analogue to the Hille-Yosida-Phillips Theorem, see [5, Theorem 13, p.624] or [6, Section 12.3]. Our other results assume either that X^* is uniformly convex or that X and X^* are uniformly convex. These results consist of sufficient conditions for an operator in X to determine a nonexpansive semi-group in certain ways, as well as some continuity properties of derivatives of nonexpansive semi-groups. For other nonlinear analogues to the theory of linear semi-groups, see [2], [3], [4], [7], [8], [10], [11], and [12]. The main tool of proof in this paper is the notion of a multivalued dissipative operator as used by Komura in [8]. For the connection between linear dissipative

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operators and linear contraction semi-groups, see [9] by Lumer and Phillips. Section 1 contains the statements of the main results and a discussion of them. Section 2 contains two lemmas about dissipative operators, and Section 3 contains the proofs of the main results.

1. The main results.

Definition 1. The <u>duality map</u> F from X into X^* is defined by $Fx = \{x^* : ||x^*|| = ||x|| \text{ and } \langle x, x^*\rangle = ||x||^2\}$. It is known that F is single valued if X^* is strictly convex, and that F is uniformly continuous on bounded sets if X^* is uniformly convex, see [7, Lemma 1.2].

By a <u>multivalued operator in X</u>, we mean a transformation A from a subset of X into the collection of all subsets of X. We (somewhat loosely) include the operators (i.e., single valued operators) among the multivalued operators.

A multivalued operator A in X is said to be <u>dissipative</u> if

$$\text{Re}\,\langle x' - y', f\rangle \leq 0$$

for $x, y \in D(A)$, $x' \in Ax$, $y' \in Ay$, and $f \in F(x - y)$.

Remark. If $\{T(t);\ t \geq 0\}$ is a nonexpansive semi-group, then both the weak and the strong infinitesimal generators of $\{T(t)\}$ are dissipative. To see this, let $t > 0$, $x, y \in X$, and $f \in F(x - y)$. Then

$$\text{Re}\,\langle (T(t)x - x) - (T(t)y - y), f\rangle =$$
$$\text{Re}\,\langle T(t)x - T(t)y, f\rangle - ||x - y||^2 \leq 0.$$

Lemma 1. Suppose A is a multivalued dissipative operator in X. Then $I - A$ has a single valued nonexpansive ''inverse'' defined on the ''range''. $R(I - A)$ of $I - A$:

$$R(I - A) = \{y\colon y \in x - Ax \text{ for some } x \in D(A)\},$$
$$(I - A)^{-1}y = \{x \in D(A)\colon y \in x - Ax\}.$$

If $R(I - A) = X$, then Ax is a closed convex set for each x in $D(A)$.

Proof. To prove the first statement, let $x, y \in D(A)$, $x' \in Ax$, $y' \in Ay$, $z = x - x'$, $w = y - y'$, and $f \in F(x - y)$. Then

$$||z - w||\cdot||x - y|| \geq \mathrm{Re}\langle z - w, f\rangle =$$
$$||x - y||^2 - \mathrm{Re}\,\langle x' - y', f\rangle \geq ||x - y||^2.$$

Thus, $||x - y|| \geq ||z - w||$.

To prove the second statement, notice that the operator A^* defined on $D(A)$ by letting A^*x be the closed convex hull of Ax is also dissipative, and thus cannot properly extend A if $R(I - A) = X$.

Definition 2. If X is uniformly convex and A is a multivalued dissipative operator in X such that $R(I - A) = X$, then we define T_A, the <u>trace of A</u>, on $D(A)$ by letting T_Ax be the point of Ax which is nearest the origin. An operator B in X is said to be a <u>trace operator</u> if it is the trace of such an operator A; B is said to be a <u>maximal trace operator</u> if B is a trace operator and no trace operator properly extends B.

Theorem 1. Suppose X^* is uniformly convex and A is a densely defined multivalued dissipative operator in X such that $R(I - A) = X$. Then there is a unique nonexpansive semi-group $\{T(t);\ t \geq 0\}$ such that:

i) If $x \in D(A)$, then $T(t)\,x \in D(A)$

a.e. on $[0, \infty)$, $T(\cdot)x$ is strongly differentiable a.e. on $[0, \infty)$, and

$$(d/dt)\, T(t)x \in A\, T(t)\, x$$

a.e. on $[0, \infty)$.

ii) If $x \in D(A)$, then $(d/dt)\, T(t)x$ is Bochner integrable on bounded intervals, and

$$T(t)\, x = x + \int_0^t (d/d\xi)\, T(\xi)x\, d\xi$$

for $t \geq 0$.

Furthermore, it is true that if $x \in D(A)$ and $y \in Ax$, then

$$\|(d/dt)\, T(t)x\| \leq \|y\|$$

a.e. on $[0, \infty)$.

Theorem 2. Suppose X and X^* are uniformly convex and B is a densely defined trace operator in X. Then there is unique nonexpansive semi-group $\{T(t);\, t \geq 0\}$ whose strong infinitesimal genreator is an extension of B. Furthermore, if $x \in D(B)$, then $T(t)x \in D(B)$ for $t \geq 0$, and $BT(\cdot)x$ is strongly continuous from the right on $[0, \infty)$. If $R(I - B) = X$ (which makes B a trace operator), then B is the strong and the weak infinitesimal generator of $\{T(t)\}$, and $BT(\cdot)x$ is weakly continuous on $[0, \infty)$ for $x \in D(B)$.

Theorem 3. If X is a Hilbert space, and B is a densely defined operator in X, then B is the strong infinitesimal generator of a nonexpansive semi-group if and only if B is a maximal trace operator.

Remarks. Theorem 1 is primarily a generalization (somewhat trivial) and a sharpening (nontrivial) of Komura's Theorem 4 in [8]. The semi-group $\{T(t)\}$ is constructed by Komura's method, but we show that the differential equation

$$(d/dt)\ T(t)x \in A\ T(t)x$$

is satisfied in a much stronger sense than that established by Komura.

Theorem 2 shows that a densely defined trace operator B is almost as good as an infinitesimal generator; not only is there a unique nonexpansive semi-group $\{T(t)\}$ whose strong infinitesimal generator extends B, but

$$T(t)x = x + \int_0^t B\,T(\xi)\,x\ d\xi\ ,$$

$$D_r T(t)x = B\ T(t)\ x$$

for $x \in D(B)$ and $t \geq 0$, where D_r denotes the strong right derivative. The last statement of Theorem 2 partly clears up points raised by Browder [2, p. 870] and Kato [7, Remark 2 after Theorem 3].

The Hille-Yosida-Phillips Theorem for linear semi-groups characterizes semi-group generators; whereas our Theorem 3 characterizes densely defined generators (in Hilbert space at that), and we do not know that every nonexpansive semi-group has a densely defined strong infinitesimal generator. Nevertheless, Theorem 3 is the only theorem of this type that we know of. It is somewhat interesting when applied to the one dimensional real case: a nonincreasing function B from the reals into the reals is the strong infinitesimal generator of a nonexpansive semi-group if and only if

$$|B(x)| = \min\left(|B(x-)|,\ |B(x+)|\right)$$

for all real x.

2. Lemmas on dissipative operators.

Lemma 2. Suppose Y is a Banach space and E is a multivalued dissipative operator in Y such that $R(I - E) = Y$. Let G denote the duality map from Y into Y^*. Suppose:

i) $\{u_n\} \subset D(E)$,

ii) $x_n \in E\, u_n$ for each n,

iii) $\lim u_n = u$, $\text{w-lim}\, x_n = x$, and

iv) if $v \in Y$, then there exists $\{f_n\} \subset Y^*$ such that $f_n \in G(v - u_n)$ for each n and $\lim f_n \in G(v - u)$.

Then $u \in D(E)$ and $x \in E\, u$. If Y^* is uniformly convex, then (iv) is superfluous.

Proof. The proof requires only obvious modifications of the proof of [7, Lemma 2.5].

Lemma 3. Suppose X^* is uniformly convex, and let $Y = L_2([0, 1], X)$, the space of (equivalence classes of) strongly measurable functions f from $[0, 1]$ into X such that $||f(\cdot)||^2$ is integrable, with

$$||f||_2 = [\int_0^1 ||f(t)||^2\, dt]^{1/2} .$$

Then Y is reflexive, the duality map G from Y into Y^* is single valued, and Y^* is isometrically isomorphic to $L_2([0, 1], X^*)$ with the action

$$\langle f, g\rangle = \int_0^1 \langle f(t), g(t)\rangle\, dt$$

of $L_2([0, 1], X^*)$ on Y.

Now suppose A is a multivalued dissipative operator in X such that $R(I - A) = X$. Define the multivalued operator E in Y by

$D(E) = \{f \in Y : f(t) \in D(A)$ a.e. and there exists $g \in Y$ such that $g(t) \in Af(t)$ a.e.$\}$,

$Ef = \{g \in Y : g(t) \in Af(t)$ a.e.$\}$.

Then E is a multivalued dissipative operator in Y, and $R(I - E) = Y$. Let $C([0, 1], X)$ denote the space of continuous functions from $[0, 1]$ into X. If

i) $\{\alpha_n\} \subset C([0, 1], X) \cap D(E)$,

ii) $\beta_n \in E\,\alpha_n$ for each n, and

iii) $\{\alpha_n\}$ converges uniformly to α and $\{\beta_n\}$ converges weakly in Y to $\beta \in Y$,

then $\alpha \in D(E)$ and $\beta \in E\,\alpha$.

Proof. If follows from [1, Theorem 3.2] and the appendix of [8] that Y is reflexive and that Y^* has the representation claimed. Consider the duality map G as being from Y into $L_2([0, 1], X^*)$. Let $f \in Y$ and $g \in Gf$. Then

$$||f||_2^2 = \int_0^1 \langle f(t), g(t)\rangle \, dt$$

$$\int_0^1 ||f(t)||\cdot||g(t)||dt \leq ||f||_2||g||_2 = ||f||_2^2 .$$

Therefore,

$$\int_0^1 ||f(t)||\cdot||g(t)||dt = ||f||_2||g||_2, \quad ||g||_2 = ||f||_2 ,$$

so that $\|g(t)\| = \|f(t)\|$ a.e. Therefore $\langle f(t), g(t)\rangle = \|f(t)\|^2$ a.e., so that $g(t) \in F\, f(t)$ a.e.

It now follows readily that the multivalued operator E is dissipative. The fact that $R(I - E) = Y$ follows from the observation that if $g \in Y$, then the function f defined by $f(t) = (I - A)^{-1} g(t)$ is also in Y, and is in $D(E)$.

To prove the last statement of this lemma, suppose that $\{\alpha_n\} \subset C([0, 1], X)$, and that $\{\alpha_n\}$ converges uniformly to α on $[0, 1]$. Let

$$k = \sup\{\|\alpha_n(t)\|: t \in [0, 1], n = 1, 2, 3, \ldots\}.$$

If $v \in Y$, then the sequence $\{\alpha_n - v\}$ is pointwise dominated by $k + \|v(\cdot)\|$; thus, so is the sequence $\{G(\alpha_n - v)\}$. Also, since $\{\alpha_n - v\}$ converges pointwise to $\alpha - v$, $G(\alpha_n - v)\}$ converges pointwise to $G(\alpha - v)$ and thus converges strongly in $L_2([0, 1], X^*)$ to $G(\alpha - v)$. The rest follows immediately from Lemma 2.

3. Proof of the main theorems.

Proof of Theorem 1. First, we prove that there is at most one nonexpansive semi-group satisfying (i) and (ii). If $\{S(t)\}$ and $\{T(t)\}$ are two such semi-groups, and they agree on $D(A)$, then they agree everywhere. Let $x \in D(A)$, $f(t) = \|T(t)x - S(t)x\|^2$. Then f is absolutely continuous, and by [7, Lemma 1.3],

$$f'(t) = 2\mathrm{Re}\,\langle (d/dt)T(t)x - (d/dt)S(t)x, F(T(t)x - S(t)x)\rangle$$

a.e., so that $f'(t)$ is nonpositive a.e., and thus $f(t) = 0$ for $t \geq 0$.

The following claim can be established by an argument almost exactly like Komura's argument for Theorem 4 of [8]. Komura's theorem is for a Hilbert space, but the argument generalizes; to see how the continuity of the duality map can be used as a substitute for the Hilbert space structure, see Kato's proof of [7, Lemma 4.3].

There is a nonexpansive semi-group $\{T(t);\ t \geq 0\}$ having the following properties. Let $x \in D(A)$, $y \in Ax$. There are two sequences $\{f_n\}$, $\{g_n\}$ in $C([0, 1], X)$ each converging uniformly to $T(\cdot)\, x$ on $[0, 1]$ such that:

i) $g_n(t) \in D(A)$ for $0 \leq t \leq 1$ and $n = 1, 2, \ldots,$

ii) $f_n{}'(t) \in A\, g_n(t)$ for $0 \leq t \leq 1$ and $n = 1, 2, \ldots,$

iii) $f_n{}'$ is continuous on $[0, 1]$ for $n = 1, 2, \ldots,$ and

iv) $\|f_n{}'(t)\| \leq \|y\|$ for $0 \leq t \leq 1$ and $n = 1, 2, \ldots$

(To make the connection with Komura's argument for [8, Theorem 4], let $f_n(t) = T_t^{(n)} x_n$, $g_n(t) = (I - (1/n)A)^{-1} f_n(t)$.)

The sequence $\{f_n{}'\}$ is bounded in $Y = L_2([0, 1], X)$, and thus some subsequence $\{f_{m_n}{}'\}$ converges weakly to an element β of Y. Let $\alpha_n = g_{m_n}$, $\beta_n = f_{m_n}{}'$, $\alpha = T(\cdot)\, x$. Then $\{\alpha_n\} \subset D(E) \cap C([0, 1], X)$, $\beta_n \in E\, \alpha_n$ for each n, $\{\alpha_n\}$ converges uniformly to α, and $\{\beta_n\}$ converges weakly to β. Therefore, $\alpha \in D(E)$ and $\beta \in E\, \alpha$. Also, $\|\beta(t)\| \leq \|y\|$ a.e., since the set of all functions in Y satisfying this condition is convex and strongly closed, thus weakly closed.

If $x^* \in X^*$, $0 \leq t \leq 1$, and $g(\xi) = \chi_{[0,t]}(\xi) x^*$ for $0 \leq \xi \leq 1$, then $g \in L_2([0, 1], X^*)$, and

$$\langle f_{m_n}(t) - f_{m_n}(0),\ x^* \rangle = \int_0^1 \langle \beta_n(\xi),\ g(\xi) \rangle \, d\xi \ ,$$

$$\langle \alpha(t) - \alpha(0),\ x^* \rangle = \int_0^t \langle \beta(\xi),\ x^* \rangle \, d\xi \ ,$$

$$\alpha(t) = \alpha(0) + \int_0^t \beta(\xi) \, d\xi \ .$$

Thus,

$$\alpha'(t) = (d/dt)\ T(t)x = \beta(t) \in AT(t)\ x,$$

and

$$||(d/dt)\ T(t)x|| = ||\beta(t)|| \le ||y||$$

a.e. on $[0, 1]$. This argument could have been given for any bounded interval, and hence the conclusions hold on $[0, \infty)$.

Proof of Theorem 2. Suppose B is the trace of the densely defined multivalued dissipative operator A in X, with $R(I - A) = X$. Let $\{T(t),\ t \ge 0\}$ denote the nonexpansive semi-group associated with A by Theorem 1 and let A_0 denote the strong infinitesimal generator of $\{T(t)\}$. We will show that $A_0 \supset B$.

Let $x \in D(B) = D(A)$. Then by Theorem 1, $||(d/dt)\ T(t)x|| \le ||Bx||$ a.e. Let $N \subset [0, 1]$ be a set of measure zero such that $T(t)x \in D(A)$, $(d/dt)\ T(t)x \in AT(t)x$, and $||(d/dt)\ T(t)x|| \le ||Bx||$ for $t \in M = [0, 1] \setminus N$. Let $\{t_n\} \subset M$ be such that $t_{n+1} < t_n$ for all n and $\lim t_n = 0$. Let $f(t) = T(t)x$ for $t \ge 0$. Then $\{f'(t_n)\}$ is a bounded sequence in X and thus has a subsequence $\{f'(t_{m_n})\}$ which converges weakly to some element z of X. Then by Lemma 2, $z \in Ax$.

Thus,

$$\|Bx\| \leq \|z\| \leq \liminf \|f'(t_{m_n})\| \leq$$

$$\limsup \|f'(t_{m_n})\| \leq \|Bx\| .$$

Thus,

$$\lim \|f'(t_{m_n})\| = \|Bx\| = \|z\| ,$$

so that

$$z = Bx = \lim f'(t_{m_n}) .$$

Thus,

$$\lim_{(t \to 0, t \in M)} (d/dt)T(t)x = Bx.$$

This, together with (ii) of Theorem 1, shows that $x \in D(A_0)$ and $A_0 x = Bx$.

Now let $x \in D(B) = D(A)$ and $0 \leq t < 1$. Let $\{t_n\} \subset M$ be such that $t_{n+1} < t_n$ for all n and $\lim t_n = t$. Again, let $f(\xi) = T(\xi)x$ for $0 \leq \xi \leq 1$. Then $\{f'(t_n)\}$ is a bounded sequence in X and thus has a subsequence $\{f'(t_{m_n})\}$ which converges weakly to an element v of X. By Lemma 2, $f(t) \in D(A) = D(B)$, and $v \in A\, f(t)$. The argument of the preceeding paragraph shows that $v = BT(t)x = A_0 T(t)x$.

Now that we know that $T(t)x \in D(B)$ for $x \in D(B)$ and $0 \leq t < 1$, the same argument shows that $BT(\cdot)x$ is strongly continuous from the right on $[0, 1)$.

The interval $[0, 1]$ could be replaced with any bounded interval in this argument, so if $x \in D(B)$, then $T(t)x \in D(B)$ for all $t \geq 0$, and $BT(\cdot)x$ is strongly continuous from the right on $[0, \infty)$.

If $R(I - B) = X$, then B has no proper dissipative extension, and thus is the strong and weak infinitesimal generator of $\{T(t)\}$. The fact that $BT(\cdot)x$ is weakly continuous on $[0, \infty)$ for $x \in D(B)$ in this case was proved by Kato [7, Lemma 4.5].

Proof of Theorem 3. Now suppose that X is a Hilbert space. Suppose that B is a densely defined maximal trace operator. Let $\{T(t); t \geq 0\}$ be the nonexpansive semi-group determined by B, and let A_0 denote the strong infinitesimal generator of $\{T(t)\}$. By [8, Theorem 2], A_0 has an extension to a multivalued dissipative operator A such that $R(I - A) = X$. The semi-group associated with A by Theorem 1 is $\{T(t)\}$, see [8, Theorem 5]. By the argument of Theorem 2, A_0 extends the trace T_A of A. But $D(T_A) = D(A) \supset D(A_0) \supset D(T_A)$, and therefore $A_0 = T_A \supset B$. Since B is a maximal trace operator, $B = A_0$.

Now suppose that $\{T(t); t \geq 0\}$ is a nonexpansive semi-group with densely defined strong infinitesimal generator A_0. The argument of the preceeding paragraph shows that A_0 is a trace operator. Suppose B is a trace operator which extends A_0, say $B = T_A$, where A is a multivalued dissipative operator in X such that $R(I - A) = X$. Let $\{S(t); t \geq 0\}$ be the nonexpansive semi-group associated with A by Theorem 1, and let B_0 denote the strict infinitesimal generator of $\{S(t)\}$. Again, by the argument for Theorem 2, we have $B_0 \supset T_A = B \supset A_0$. The uniqueness argument given for Theorem 1 shows that $\{S(t)\}$ and $\{T(t)\}$ agree on $D(A)$, which is dense, so $\{S(t)\} = \{T(t)\}$, and $B_0 = A_0$. Therefore $B = A_0$, and A_0 is a maximal trace operator.

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